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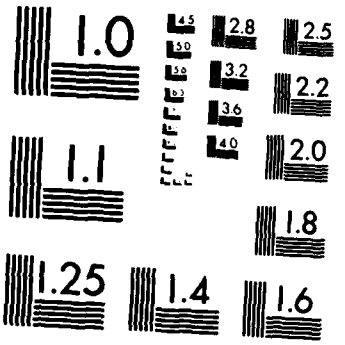

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Determining the Stability of a Mean Estimate from Correlated Samples By Use of Linear Prediction

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Preface

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The estimate of the mean of a set of N data samples of a stationary random process has unknown statistical quality if the samples are statistically dependent on each other, with unknown covariance. In order to determine the stability of a mean estimate, a second-order statistic, such as the covariance function, must also be estimated; more specifically, the origin value of the spectrum must be estimated. Three procedures for accomplishing this are presented; it is recommended that the linear predictive technique be employed because it has better resolution and stability properties and can be applied to shorter data segments.

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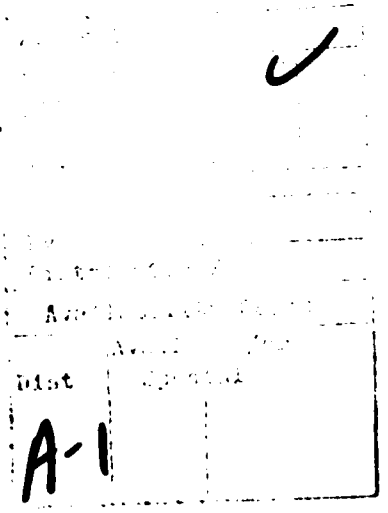

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

N	Number of data samples
t	Time
x(t)	Stationary random process
overbar	Ensemble average
m	Mean of x(t), (1)
$R_x(\tau)$	Covariance of x(t), (2)
f	Frequency
$G_x(f)$	Spectrum of x(t), (3)
Δ	Sampling increment in time, (4)
$\hat{m}$	Estimate of m, (6)
w(n)	Weights used in average, (6)
var	Variance
σ_x^2	Variance of x(t)
v	Variance of $\hat{m}$, (11)
$\phi(k)$	Autocorrelation of w(n), (12)
K	Number of pieces of data, (15)
D	Size of each piece, (15),(16)

LIST OF SYMBOLS (Cont'd)

$\hat{m}_k$	Mean estimate from k-th piece
$\hat{v}$	Estimate of v, (18),(28),(32)
$\tilde{G}_x(f)$	Spectrum of sampled process, (21)
$X(\ell)$	Fourier coefficient, (24)
$\delta_{\ell 0}$	1 if $\ell = 0$; 0 otherwise
L	Spectral region to be averaged, (28)
E_o	Minimum mean square error, (30)
$\tilde{a}_n$	Predictive filter coefficient
$\hat{G}_x(0)$	Estimate of spectrum at origin, (31)
$\hat{R}_x(k\Delta)$	Estimate of covariance, (31)
F_o	Minimum average error, (31)
p	Predictive filter order, (31)
M, 1, Q	Auxiliary matrices, (A-9)
S	Sum of covariance samples, (A-10)

DETERMINING THE STABILITY OF A MEAN ESTIMATE FROM CORRELATED SAMPLES BY USE OF LINEAR PREDICTION

INTRODUCTION

When a collection of N data samples of a stationary process are taken, a common method of estimating the mean of the random process is to use the sample mean of the available data. If the data samples are taken with a time increment large enough that they are linearly independent of each other, that is, uncorrelated samples, then the variance of the mean estimate is equal to the actual (unknown) variance of the process divided by N. In this case, the quality of the mean estimate can be approximated by also estimating the process variance, that is, by computing the sample variance of the available data.

However, when the time increment is small enough that the data samples are statistically dependent on each other, the variance of the mean estimate depends on the (unknown) covariance of the given process. Thus, in order to determine the stability of a first-order statistic (mean), we need information on a second-order statistic (covariance or spectrum). Since this latter information is unknown, we also need to estimate it from the available data. Three procedures for accomplishing this goal will be discussed here.

FUNDAMENTAL EQUATIONS

DEFINITIONS

Suppose that real stationary random process x(t) has mean

$$m = \overline{x(t)} \tag{1}$$

and covariance function

$$R_x(\tau) = \overline{[x(t) - m][x(t - \tau) - m]} \ . \tag{2}$$

(An overbar denotes an ensemble average.) The spectrum of process x(t) is*

$$G_x(f) = \int d\tau \ \exp(-i2\pi f\tau) \ R_x(\tau) \ . \tag{3}$$

Neither R_x nor G_x contain any information about the mean value, m, of the process. We presume that m, $R_x(\tau)$, and $G_x(f)$ are all unknown.

SAMPLING PROCEDURE

Suppose that N data samples,

$$x\{n\Delta\} \quad \text{for} \quad 1 \leq n \leq N \ , \tag{4}$$

---

*Integrations and summations without limits are over the complete range of nonzero integrands and summands, respectively.

are taken of process x(t), where time increment Δ might be small enough that the samples are correlated; that is,

$$R_x(n\Delta) \neq 0 \quad \text{for} \quad n = \pm 1, \pm 2, \ldots . \tag{5}$$

We estimate the actual (unknown) mean m according to

$$\hat{m} = \frac{1}{N} \sum_{n=1}^{N} x(n\Delta) = \sum_{n} w(n)\, x(n\Delta) , \tag{6}$$

where (for notational convenience) weights

$$w(n) = \begin{cases} 1/N & \text{for } 1 \leq n \leq N \\ 0 & \text{otherwise} \end{cases} . \tag{7}$$

Now, the estimate $\hat{m}$ in (6) is an unbiased estimate of m, because

$$\bar{\hat{m}} = \frac{1}{N} \sum_{n=1}^{N} \overline{x(n\Delta)} = m , \tag{8}$$

upon use of (1). But how good an estimate is $\hat{m}$ of m; that is, what is the standard deviation of random variable $\hat{m}$? This information is needed in order to establish reasonable confidence limits about a particular value of an estimate, $\hat{m}$.

To illustrate the wide range of possible values for the stability of random variable $\hat{m}$, consider the two limiting cases where (a) the samples

$\{x(n\Delta)\}$ are all linearly independent (uncorrelated) of each other, or alternatively, (b) completely statistically dependent on each other. As shown below, the variance of $\hat{m}$ is

$$\text{var}(\hat{m}) = \begin{cases} \sigma_x^2/N & \text{for (a)} \\ \sigma_x^2 & \text{for (b)} \end{cases}, \tag{9}$$

where σ_x^2 is the (unknown) variance of process x(t). These extreme values differ by a factor of N, the number of available data samples. In order to determine just where in this range the variance of estimate $\hat{m}$ lies for a particular application, more information must be extracted from the available data $\{x(n\Delta)\}$.

VARIANCE OF ESTIMATE $\hat{m}$

To determining the variance of $\hat{m}$, consider the difference random variable

$$\hat{m} - m = \frac{1}{N} \sum_{n=1}^{N} [x(n\Delta) - m] = \sum_{n} w(n) [x(n\Delta) - m] . \tag{10}$$

Then the variance of mean estimate $\hat{m}$ is

$$v = \text{var}(\hat{m}) = \overline{(\hat{m} - m)^2} =$$

$$= \sum_{n\ p} w(n)\ w(p)\ \overline{[x(n\Delta) - m][x(p\Delta) - m]} =$$

$$= \sum_{n\ p} w(n)\ w(p)\ R_x(n\Delta - p\Delta) = \sum_{k} \phi(k)\ R_x(k\Delta) , \tag{11}$$

where we used covariance (2) and defined

$$\phi(k) = \sum_{n} w(n)\, w(n-k) \tag{12}$$

as the autocorrelation of weight sequence $\{w(n)\}$ in (6). For the equi-weight sequence of (7), this function is depicted in figure 1.

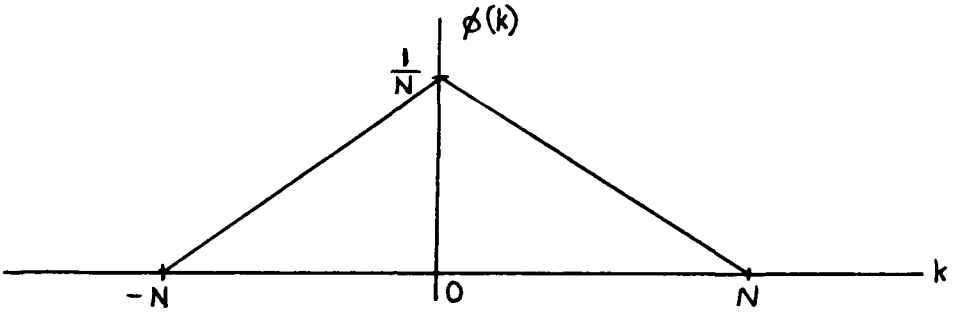

Figure 1. Autocorrelation of Weight Sequence (7)

Now $N\Delta$ is the total observation time covered by the samples $\{x(n\Delta)\}$; see (4). Suppose that

$$N\Delta \gg \text{effective extent of } R_x(\tau) \ ; \tag{13}$$

the right hand side of (13) is also called the correlation (actually covariance) time of process x(t). If (13) is satisfied, then the variance in (11) simplifies to

$$v = \text{var}(\hat{m}) \cong \sum_{k} \phi(0)\, R_x(k\Delta) = \frac{1}{N} \sum_{k} R_x(k\Delta) \ . \tag{14}$$

Thus, when the observation time is much larger than the correlation time of the process, the variance of $\hat{m}$ depends only on the sum of the sampled covariance values of the process x(t). (Result (9) follows easily from (11) and figure 1, since $R_x(0) = \sigma_x^2$.)

However, since covariance $R_x(\tau)$ is unknown, we cannot determine the right-hand side of (14), meaning that we do not know the quality of estimate $\hat{m}$. In the next section, we will address three procedures for determining the variance v of estimate $\hat{m}$.

PROCEDURES FOR ESTIMATION

In this section, we will present three procedures for determining the variance of mean estimate $\hat{m}$ and discuss some of their limitations.

MULTIPLE PIECES

This first procedure, which is probably the most obvious one, consists of dividing the available N data points in (4) into K abutting pieces of data, each of size D, where

$$N = K\,D\ . \tag{15}$$

For the k-th piece, we estimate the mean of x(t) according to

$$\hat{m}_k = \frac{1}{D} \sum_{d=1}^{D} x\big(d\Delta + (k-1)D\Delta\big) \quad \text{for} \quad 1 \le k \le K\ . \tag{16}$$

Then the average of these K sub-estimates is simply

$$\frac{1}{K} \sum_{k=1}^{K} \hat{m}_k = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{D} \sum_{d=1}^{D} x\big(d\Delta + (k-1)D\Delta\big) = \frac{1}{N} \sum_{n=1}^{N} x(n\Delta) = \hat{m}\ , \tag{17}$$

the overall estimate already considered above in (6).

Additionally, we now have the option of looking at the variability of the K random variables $\{\hat{m}_k\}$ in (16) in order to see how useful they can be in

estimating variance v in (11)-(14). Consider the quantity (for $K \geq 2$)

$$\hat{v} \equiv \frac{1}{K(K-1)} \sum_{k=1}^{K} (\hat{m}_k - \hat{m})^2 , \tag{18}$$

which can be formed from the K mean estimates in (16). It is shown in appendix A that $\hat{v}$ in (18) is an unbiased estimate of variance v in (14), under a condition on unknown covariance R_x. Specifically, we find that

$$\bar{\hat{v}} = v = \text{var}(\hat{m}) , \tag{19}$$

if

$$D\Delta \gg \text{effective extent of } R_x(\tau) . \tag{20}$$

Since $D\Delta$ is the individual segment length of each data piece used in (16), constraint (20) is a rather restrictive condition. It states that each segment cannot be too short; therefore, the number of pieces, K, in (15) and (18) cannot be too large. However, since the stability of estimate $\hat{v}$ depends inversely on K (see (A-13)), satisfaction of (20) may not allow for a very stable estimate $\hat{v}$ via (18).

What we are doing here, by means of (15)-(18), is breaking the available data record into a number of pieces which yield essentially independent estimates of mean m. The segment length, $D\Delta$, being longer than the effective extent of covariance $R_x(\tau)$ guarantees this property. If this condition, (20), is not met, the linear taper in (A-3) affects the sum there in an unknown way and can severely bias the value of $\overline{\hat{m}_1^2}$. Also, (20) is a much

more restrictive condition than (13), since $D\Delta$ is an individual segment length, while $N\Delta$ is the total observation interval. It would be very worthwhile to employ a technique which is not subject to the restrictive condition (20); however, we must be willing to accept condition (13).

If this multiple piece technique is employed and condition (20) is satisfied, then the square root of (18), $\sqrt{\hat{v}}$, can be used as an estimate of the standard deviation of mean estimate $\hat{m}$ in (6). This is similar to the approach in [1, appendix D, (D-5)], but done more rigorously here. More details are furnished in appendix A.

PERIODOGRAM

The variance v of mean estimate $\hat{m}$ in (6) is given by (14), when condition (13) is satisfied. Now the spectrum of the ac component of the sampled process $\{x(n\Delta)\}$ is defined as

$$\tilde{G}_x(f) = \Delta \sum_k R_x(k\Delta) \exp(-i2\pi fk\Delta) , \tag{21}$$

where R_x is the covariance of x; see (2). This quantity is related to the spectrum of the ac component of the continuous process x(t), namely $G_x(f)$ in (3), according to

$$\tilde{G}_x(f) = \sum_n G_x\left(f - \frac{n}{\Delta}\right) ; \tag{22}$$

that is, $\tilde{G}_x(f)$ is the aliased version of $G_x(f)$. Reference to (14) and (21) reveals that variance

$$v = \frac{1}{N\Delta}\,\tilde{G}_x(0) \ . \tag{23}$$

Thus, if we can estimate the origin value of the spectrum $\tilde{G}_x(f)$ of sampled process $\{x(n\Delta)\}$, we will have an estimate of v.

Consider, then, the discrete Fourier transform of the given data sequence:

$$X(\ell) = \sum_{n=0}^{N-1} x((n+1)\Delta)\,\exp(-i2\pi n\ell/N) \quad \text{for } 0 \le \ell \le N-1 \ . \tag{24}$$

Its average power, at any frequency index ℓ, is

$$\overline{|X(\ell)|^2} = \sum_{n,p=0}^{N-1} [m^2 + R_x(n\Delta - p\Delta)]\,\exp(-i2\pi(n-p)\ell/N) =$$

$$= m^2N^2\delta_{\ell 0} + N\sum_{k=-N}^{N}\left(1 - \frac{|k|}{N}\right) R_x(k\Delta)\,\exp(-i2\pi k\ell/N) \ , \tag{25}$$

by use of (2). Then for $\ell \neq 0$, if condition (13) on observation time $N\Delta$ is observed, we have the approximation

$$\overline{|X(\ell)|^2} \cong N\sum_{k} R_x(k\Delta)\,\exp(-i2\pi k\ell/N) = \frac{N}{\Delta}\,\tilde{G}_x\left(\frac{\ell}{N\Delta}\right) \ , \tag{26}$$

upon use of (21). In particular

$$\frac{1}{N^2}\,\overline{|X(\ell)|^2} = \frac{1}{N\Delta}\,\tilde{G}_x\left(\frac{\ell}{N\Delta}\right) \quad \text{for } \ell \neq 0 \ . \tag{27}$$

This relation suggests the following procedure for estimating the variance v, as given by (23): compute and plot the periodogram $|X(\ell)|^2$ for $0 \le \ell \le N - 1$, as obtained from (24); observe the region where these (noisy) values are approximately equal to the origin values, excluding $\ell = 0$; average these magnitude-squared values over this region. If this region is $1 \le \ell \le L$, say, then the estimate of variance is, guided by (23) and (27),

$$\hat{v} = \frac{1}{N^2} \frac{1}{L} \sum_{\ell=1}^{L} |X(\ell)|^2 . \tag{28}$$

The value of the discrete Fourier transform at the origin is of no use for variance estimation since $X(0) = N \hat{m}$, which involves the sample mean. Also since the data $\{x(n\Delta)\}$ have been presumed real here, then $X(N - \ell) = X^*(\ell)$, and the spectral region (negative frequencies) near $\ell = N - 1$ contains no new information.

This procedure also employed condition (13), in the process of deriving (26). However, the most important limitation of this ad hoc procedure is the arbitrary decision about the region L in frequency that should be used in average (28). The wider the region, the greater the stability in $\hat{v}$; however, too wide a selection will result in a biased estimate of variance v.

LINEAR PREDICTION

This procedure will be heavily based upon the material in [3] on linear predictive spectral estimation, which the reader must be familiar with. We first observe from (21) that

$$\frac{1}{\Delta}\,\tilde{G}_x(0) = \sum_k R_x(k\Delta)\ . \tag{29}$$

Also, from [3; (14)],

$$\frac{1}{\Delta}\,\tilde{G}_x(0) = \frac{E_o}{\left|\sum_{n=0}^{\infty} \tilde{a}_n\right|^2}\ , \tag{30}$$

where E_o is the minimum mean square prediction error [3; (8)] and filter coefficient $\tilde{a}_o = -1$ [3; (7)]. Thus, the variance v in (23) can be estimated if we can estimate the quantities on the right-hand side of (30).

For unknown process covariance, we refer to [3; (143) and (108)] to obtain estimate

$$\frac{1}{\Delta}\,\hat{G}_x(0) = \sum_k \hat{R}_x(k\Delta) = \frac{F_o}{\left|1 - \sum_{n=1}^{p} \tilde{a}_n\right|^2}\ , \tag{31}$$

where F_o is the minimum average magnitude-squared error as given by [3; (137) and (141)], p is the predictive filter order that converts the predictive error to a white process, and $\{\tilde{a}_n\}_1^p$ are the p-th order

predictive filter coefficients. A comparison of (23) and (31) then leads to variance estimate

$$\hat{v} = \frac{1}{N} \frac{F_o}{\left| 1 - \sum_{n=1}^{p} \tilde{a}_n \right|^2}, \tag{32}$$

in terms of the linear predictive filter coefficients and associated error. The selection of order p is somewhat subjective, but can be accomplished by use of various information criteria [4].

In order to avoid the deleterious effects of a large mean component m in data $\{x(n\Delta)\}$ on the filter coefficients and the origin spectral estimate in (31), the sample mean, $\hat{m}$, must be subtracted from the given data before the linear predictive approach is applied.

SUMMARY AND DISCUSSION

Sectioning of the available data into a number of disjoint pieces requires that the length of each individual piece be much longer than the correlation time of the process. For a given total data record, this limits the number of pieces that can be employed to get a stable estimate of the variance of the mean estimate. Accordingly this technique is not recommended unless a very large data record is available and stability is not an issue.

The periodogram technique requires an ad hoc (eyeball) decision as to the region near the frequency origin where the average of bin powers should be conducted. Other than this, it only requires that the total data record length $N\Delta$ be much larger than the correlation time of the process; this condition is a reasonable one and will be required of any technique attempting to extract a stable estimate of the variance of $\hat{m}$.

The linear predictive approach is also subject to this condition on total observation time, since $p\Delta$ is the duration of the impulse response of the linear predictive filter, and this must be short relative to $N\Delta$, in order to get stable estimates of filter coefficients $\{\tilde{a}_n\}$. However, we can utilize both a forward as well as a backward average [3] for additional stability. Also, there is an automatic indication of whether $p\Delta \ll N\Delta$, simply by observing when the predictive error saturates as the filter order increases; that is, the filter order need not be taken any larger than necessary, thereby preserving the quality of the resulting estimates.

If the process x(t) under investigation is complex, the current analyses still apply, with some generalizations. Thus, the second bracketed term in (2) for the covariance must be conjugated, and the variance in (11) must be a magnitude-squared error. The linear predictive procedures in [3, 4] were, in fact, carried out for complex processes and so apply directly.

These techniques also afford an efficient method of assigning an equivalent number, N_e, of independent samples to a given data set, (4). The variance of $\hat{m}$ is given in (14), while the variance of a set of independent samples is given by (9a). If we equate (14) to the variance which pertains to equivalent number N_e, we have

$$\frac{1}{N}\sum_{k} R_x(k\Delta) = \frac{1}{N_e} R_x(0) \ , \tag{33}$$

or

$$N_e = N \frac{R_x(0)}{\sum_{k} R_x(k\Delta)} \ . \tag{34}$$

Since the quantities on the right hand side are unknown, we replace them by estimates obtainable from the data. In particular, using the linear predictive results in (31), we are led to estimate

$$\hat{N}_e = N \frac{\hat{R}_x(0)}{F_o} \left| 1 - \sum_{n=1}^{p} \tilde{a}_n \right|^2 \ . \tag{35}$$

APPENDIX A. DERIVATIONS FOR MULTIPLE PIECES TECHNIQUE

In this appendix, we concentrate on the technique described in (15)-(18). The estimate $\hat{v}$ in (18) can be expanded, resulting in

$$\hat{v} = \frac{1}{K(K-1)} \sum_{k=1}^{K} (\hat{m}_k^2 - 2\,\hat{m}\,\hat{m}_k + \hat{m}^2) =$$

$$= \frac{1}{K(K-1)} \left(\sum_{k=1}^{K} \hat{m}_k^2 - K\,\hat{m}^2 \right) , \tag{A-1}$$

where we employed (17). The mean of the random variable in (A-1) is then

$$\bar{\hat{v}} = \frac{1}{K-1} \left(\overline{\hat{m}_1^2} - \overline{\hat{m}^2} \right) , \tag{A-2}$$

where $\hat{m}_1$ is taken as a representative mean estimate from the set $\{\hat{m}_k\}$ in (16). But from (16), the quantity

$$\overline{\hat{m}_1^2} = \overline{\left(\frac{1}{D} \sum_{d=1}^{D} x(d\Delta) \right)^2} = \frac{1}{D^2} \sum_{d,j=1}^{D} \overline{x(d\Delta)\,x(j\Delta)} =$$

$$= \frac{1}{D^2} \sum_{d,j=1}^{D} \left(m^2 + R_x(d\Delta - j\Delta) \right) = m^2 + \frac{1}{D^2} \sum_{d,j=1}^{D} R_x(d\Delta - j\Delta) =$$

$$= m^2 + \frac{1}{D} \sum_{k=-D}^{D} \left(1 - \frac{|k|}{D} \right) R_x(k\Delta) , \tag{A-3}$$

by use of (2).

Now if the number of data points, D, in each piece (16), is taken large enough that

$$D\Delta \gg \text{effective extent of } R_x(\tau) \ , \tag{A-4}$$

then (A-3) simplifies to

$$\overline{\hat{m}_1^2} \cong m^2 + \frac{1}{D} \sum_k R_x(k\Delta) \ . \tag{A-5}$$

The corresponding result for $\overline{\hat{m}^2}$ is immediately obtained by replacing D by N everywhere, getting

$$\overline{\hat{m}^2} = m^2 + \frac{1}{N} \sum_k R_x(k\Delta) \ . \tag{A-6}$$

When we employ these last two relations in (A-2), there follows

$$\overline{\hat{v}} = \frac{1}{K-1}\left(\frac{1}{D} - \frac{1}{N}\right) \sum_k R_x(k\Delta) = \frac{1}{N} \sum_k R_x(k\Delta) = v \ , \tag{A-7}$$

where we also used (15), N = KD, and (14). Thus, estimate $\hat{v}$ in (18) is an unbiased estimator of $v = \text{var}(\hat{m})$, as claimed. No Gaussian assumptions have been utilized anywhere.

To minimize the variance of random variable $\hat{v}$ itself, as given by (18), we should take K large. However, for a limited data set of size N, this will decrease D according to D = N/K. Thus, K can only be increased to the point where (A-4) would begin to be violated. If we satisfy (A-4), then we can use $\sqrt{\hat{v}}$ as an estimate of the standard deviation of $\hat{m}$. We do not need or utilize

detailed knowledge of covariance $R_x(\tau)$, except for the condition (A-4); however, this may be a very restrictive condition.

To demonstrate that large K is beneficial for estimate $\hat{v}$ in (18), we will derive its variance under the assumption that random variables $\{\hat{m}_k\}$ in (16) are independent Gaussian random variables. This is a reasonable assumption if D is large, as already presumed. (If data $\{x(n\Delta)\}_1^N$ were Gaussian, that would guarantee the Gaussian character of $\{\hat{m}_k\}$, since they are obtained via linear operations on the data; see (16).)

Using (17), we express random variable (A-1) as

$$\hat{v} = \frac{1}{K(K-1)} \left[\sum_{k=1}^{K} \hat{m}_k^2 - \frac{1}{K} \left(\sum_{k=1}^{K} \hat{m}_k \right)^2 \right] =$$

$$= \frac{1}{K(K-1)} M^T Q M , \qquad \text{(A-8)}$$

where matrices

$$M^T = [\hat{m}_1 \ \ldots \ \hat{m}_K] ,$$

$$1^T = [1 \ \ldots \ 1] ,$$

$$Q = I - \frac{1}{K} 1 \, 1^T . \qquad \text{(A-9)}$$

We already know that

$$\overline{\hat{m}_k} = m \ , \quad \mathrm{var}(\hat{m}_k) = \frac{1}{D} \sum_k R_x(k\Delta) \cong \frac{S}{D} \ , \tag{A-10}$$

from (16) and (A-5).

We now employ [2]; in particular, see (15)-(17). Then [2; (30)] gives, with identifications

$$N \to K, \ \gamma \to 1, \ \rho \to 1, \ \sigma_u = \sigma_v \to \sqrt{S/D} \ , \tag{A-11}$$

the results

$$\overline{\hat{v}} = \frac{1}{K(K-1)} (K-1) \frac{S}{D} = \frac{S}{KD} = \frac{1}{N} \sum_k R_x(k\Delta) \tag{A-12}$$

and

$$\mathrm{var}(\hat{v}) = \frac{1}{K^2(K-1)^2} (K-1) \, 2 \frac{S^2}{D^2} = \frac{2}{N^2(K-1)} \left(\sum_k R_x(k\Delta) \right)^2 \ . \tag{A-13}$$

The first result checks (A-7), as expected. The second result involves only one free parameter, namely K, and is minimized by choosing K as large as possible, but always subject to constraint (A-4).

Observe that quality ratio

$$\frac{\bar{\hat{v}}^2}{\mathrm{var}(\hat{v})} = \frac{K-1}{2}, \qquad \text{(A-14)}$$

in terms of the "number of independent pieces," K, employed in (16), and is independent of covariance $R_x(\tau)$.

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